LIMITS OF GROUPOID C*-ALGEBRAS ARISING FROM OPEN COVERS

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ABSTRACT. I. Raeburn and J. Taylor have constructed continuous-trace C^*-algebras with a prescribed Dixmier-Douady class, which also depend on the choice of an open cover of the spectrum. We study the asymptotic behavior of these algebras with respect to certain refinements of the cover and appropriate extension of cocycles. This leads to the analysis of a limit groupoid G and a cocycle σ, and the algebra $C^*(G, \sigma)$ may be regarded as a generalized direct limit of the Raeburn-Taylor algebras. As a special case, all UHF C^*-algebras arise from this limit construction.

1. INTRODUCTION

In their study of continuous-trace C^*-algebras, J. Dixmier and A. Douady [DD63], [Dix63] introduced a complete invariant classifying these algebras up to spectrum preserving Morita equivalence. If A is a continuous-trace C^*-algebra with spectrum $\mathcal{T}$, then its Dixmier-Douady invariant $\delta(A)$ is an element of the sheaf cohomology group $H^2(\mathcal{T}, \underline{\mathbb{T}})$, where $\underline{\mathbb{T}}$ denotes the sheaf of germs of continuous circle-valued functions on $\mathcal{T}$. We remark that $H^2(\mathcal{T}, \underline{\mathbb{T}})$ is naturally isomorphic to the more familiar cohomology group $H^3(\mathcal{T}, \mathbb{Z})$. Dixmier and Douady also proved that given a locally compact paracompact Hausdorff space $\mathcal{T}$, every element of $H^2(\mathcal{T}, \underline{\mathbb{T}}) \simeq H^3(\mathcal{T}, \mathbb{Z})$ is the Dixmier-Douady invariant of some continuous-trace C*-algebra with spectrum $\mathcal{T}$. This lead naturally to the study of the Brauer group, which is the set of spectrum preserving Morita equivalence classes of continuous-trace class C^*-algebras with spectrum $\mathcal{T}$ endowed with the multiplication induced by spectrum preserving tensor product. The map $\delta : \mathrm{Brauer}(\mathcal{T}) \to H^2(\mathcal{T}, \underline{\mathbb{T}})$ is in fact a group isomorphism (for a modern treatment of the subject, see [RW98]).

The original proof of the surjectivity of δ used the contractibility of the unitary group of an infinite dimensional Hilbert space as well as Zorn's lemma, and therefore it did not lead to explicit constructions. Later I. Raeburn and J. Taylor [RT85] provided a more direct approach. We shall henceforth take $\mathcal{T}$ to be compact. Given a finite open cover U of $\mathcal{T}$ and a cocycle $\nu \in Z^2(U, \underline{\mathbb{T}})$, Raeburn and Taylor constructed an algebra $A(U, \nu)$ with spectrum $\mathcal{T}$ and Dixmier-Douady invariant represented by ν. In addition, they observed that there exists a locally compact Hausdorff étale groupoid $\mathcal{G}_U$ and a circle-valued groupoid 2-cocycle $\tau \in Z^2(\mathcal{G}_U, \mathbb{T})$ such that $A(U, \nu) \simeq C^*(\mathcal{G}_U, \tau)$.

In this work we explore the asymptotic behavior of the algebras $C^*(\mathcal{G}_U, \tau)$, as one refines the cover U while retaining a certain compatibility of cocycles. This is accomplished in terms of an étale groupoid G and an appropriate cocycle σ. More precisely, we say that $\mathcal{W}$ is an intersection refinement of $\mathcal{U}$, denoted $\mathcal{U} \leq \mathcal{W}$, if there exists a cover $\mathcal{V}$ such that

$$\mathcal{W} = \mathcal{U} \cap \mathcal{V} = \{U_i \cap V_j \mid U_i \in \mathcal{U}, V_j \in \mathcal{V}\}.$$

Any sequence of open covers $\{V^{(k)}\}_{k=1}^{\infty}$ gives rise to a sequence of intersection refinements

$$W^{(0)} \leq W^{(1)} \leq W^{(2)} \leq ...$$

of $\mathcal{T}$, where $W^{(i+1)} = W^{(i)} \cap V^{(i+1)}$. We denote $G_n = \mathcal{G}_{W^{(n)}}$, and from this sequence we construct the groupoid G.

In order to carry out this construction, and in particular to define the compatibility of the cocycles, one is lead naturally to technical considerations. The naive compatibility condition, namely that the sequence $\sigma_n \in Z^2(G_n, \mathbb{T})$ be generated from pullbacks along the projection maps $G_{n+1} \to G_n$, is insufficient, and a stronger condition is needed. In fact, we are compelled to carry out a careful analysis of several associated groupoids which are not étale, ultimately leading to the étale groupoid G and limit cocycle σ.

Our main result states the following (see Theorems 5.11 and 7.2). Recall that the support of a set $S \subseteq C^*(G, \sigma)$ is the set of points in G for which there is an element of S that does not vanish as a function in $C_0(G)$.

Main Theorem. *Let G_n be a sequence of groupoids corresponding to a sequence of intersection refinements and $\sigma_n \in Z^2(G_n, \mathbb{T})$ an associated sequence of compatible cocycles. There exists a locally compact Hausdorff principal amenable étale groupoid G and a cocycle $\sigma \in Z^2(G, \mathbb{T})$ as well as a sequence of isometric $*$-homomorphisms $\varphi_n : C^*(G_n, \sigma_n) \to C^*(G, \sigma)$ such that G is the support of $\bigcup \varphi_n(C^*(G_n, \sigma_n))$.*

In the particular case when $\mathcal{T}$ is a point, $C^*(G)$ is a UHF algebra (i.e. a direct limit of finite-dimensional C^*-algebras), and moreover all UHF algebras arise from an appropriate choice of infinite intersection refinement. Furthermore, in that case we have in fact that $C^*(G)$ is the direct limit of the algebras $C^*(G_n)$.

In a more general setting the main issue is that we cannot construct maps from $C^*(G_n, \sigma_n)$ to $C^*(G_{n+1}, \sigma_{n+1})$, and moreover the set $S = \bigcup \varphi_n(C^*(G_n, \sigma_n))$ is not always dense inside $C^*(G, \sigma)$. Therefore we do not have a direct limit. However, G is the support of S, and we regard our construction as a generalized direct limit, especially in light of the work of P. Muhly and B. Solel [MS89].

Apart from presenting our main results, we study the properties of the groupoid G and several other groupoids related to covers and to sequences of refinements.

2. PRELIMINARIES

Let G be a second countable locally compact Hausdorff principal groupoid. We shall denote the unit space of G by $G^{(0)}$ and the set of composable pairs by $G^{(2)}$. We shall also denote by r and d the range and source maps, respectively, and set $G^u = \{x \in G : r(x) = u\}$ and $G_u = \{x \in G : d(x) = u\}$ for all $u \in G^{(0)}$. The groupoid G is said to be **r-discrete** if $G^{(0)}$ is open. We will say that G is an **étale** groupoid if r is a local homeomorphism (We remark that the notions of étale and r-discrete are a source of confusion in the literature. We refer the reader to [Res07] for a detailed discussion on the relations between those definitions). If G is an étale groupoid, then the family $\lambda = \{\lambda^u : u \in G^{(0)}\}$, where λ^u is counting measure on G^u, provides a continuous left Haar system for G. We will always assume that étale groupoids are endowed with the counting Haar system.

We say that $H \subseteq G$ is a **subgroupoid** if it is closed under the multiplication and inverse operations of G. We emphasize that we do not require that H and G share the same unit space.

Given two groupoids G and H, a map $\phi : G \to H$ is a **groupoid homomorphism** if for every composable pair (x, y), we have that $(\phi(x), \phi(y))$ is also composable, $\phi(xy) = \phi(x)\phi(y)$, and for every $x \in G$, $\phi(x^{-1}) = [\phi(x)]^{-1}$.

We shall say that $\sigma : G^{(2)} \to \mathbb{T}$ is a **continuous 2-cocycle** if σ is continuous and

$$\sigma(x_0 x_1, x_2)\sigma(x_0, x_1) = \sigma(x_1, x_2)\sigma(x_0, x_1 x_2)$$

for all $x_0, x_1, x_2 \in G$ such that (x_0, x_1) and (x_1, x_2) are composable. We shall say that $\sigma : G^{(2)} \to \mathbb{T}$ is a **continuous 2-coboundary** if there exists a continuous function $\mu : G \to \mathbb{T}$ such that for all $(x, y) \in G^{(2)}$,

$$\sigma(x, y) = \mu(x)\mu(y)[\mu(xy)]^{-1}.$$

The set $Z^2(G, \mathbb{T})$ of continuous 2-cocycles becomes a group when it is endowed with the operation of pointwise multiplication of $\mathbb{T}$-valued functions. The set $B^2(G, \mathbb{T})$ of all continuous 2-coboundaries is a normal subgroup, and we define the **second cohomology group** of G with coefficients in $\mathbb{T}$ to be the quotient:

$$H^2(G, \mathbb{T}) = Z^2(G, \mathbb{T})/B^2(G, \mathbb{T}).$$

Given a 2-cocycle σ, we define the $*$-algebra of functions $C_c(G, \sigma)$ to be the set of functions $C_c(G)$ endowed with the multiplication and involution

$$(f * g)(x) = \int f(xy)g(y^{-1})\sigma(xy, y^{-1})d\lambda^{d(x)}(y),$$

$$f^*(x) = \overline{f(x^{-1})\sigma(x, x^{-1})}.$$

For $f \in C_c(G, \sigma)$, set

$$\|f\|_{I,r} = \sup_{u \in G^{(0)}} \int |f|d\lambda^u, \qquad \|f\|_{I,d} = \sup_{u \in G^{(0)}} \int |f^*|d\lambda^u.$$

We define a $*$-algebra norm $\|\cdot\|_I$ on $C_c(G, \sigma)$ by the expression:

$$\|f\|_I = \max\{\|f\|_{I,r}, \|f\|_{I,d}\}.$$

A **bounded representation** of $C_c(G, \sigma)$ on a Hilbert space H is a $*$-homomorphism $\pi : C_c(G, \sigma) \to B(H)$ that is nondegenerate, continuous (when $C_c(G, \sigma)$ is endowed with the inductive limit topology and $B(H)$ has the weak operator topology) and satisfies $\|\pi(f)\| \le \|f\|_I$ for all $f \in C_c(G, \sigma)$. The **full twisted groupoid C^*-algebra** $C^*(G, \sigma)$ is the closure of $C_c(G, \sigma)$ with respect to the following C^*-norm:

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ is a bounded representation of } C_c(G, \sigma)\}.$$

Given a measure μ on $G^{(0)}$, we define a measure ν on G by $\nu = \int_G \lambda^u d\mu(u)$ and write ν^{-1} for the image of ν under inversion. Such a measure gives rise to a bounded representation Ind_μ of $C_c(G, \sigma)$ on $L^2(\nu^{-1})$: for $f \in C_c(G, \sigma)$ and $\xi \in L^2(\nu^{-1})$, set

$$(\mathrm{Ind}_\mu(f)\xi)(x) = \int_G f(xy)\xi(y^{-1})\sigma(xy, y^{-1})d\lambda^{d(x)}(y).$$

The **reduced twisted groupoid C^*-algebra** $C_r^*(G, \sigma)$ is the closure of $C_c(G, \sigma)$ with respect to the following C^*-norm:

$$\|f\|_r = \sup\{\|\mathrm{Ind}_\mu(f)\| : \mu \text{ is a Radon measure on } G^{(0)}\}.$$

It is worth noting that by a disintegration argument (see [Muh]):

$$\|f\|_r = \sup\{\|\mathrm{Ind}_\mu(f)\| : \mu \text{ is a unit point mass on } G^{(0)}\}.$$

In fact (see the comment in [Muh] following the definition of the reduced norm), if E is a saturating subset of $G^{(0)}$ (i.e. such that for every $u \in G^{(0)}$ there exists $x \in G$ such that $r(x) = u$ and $d(x) \in E$), then:

(2.0.1) $$\|f\|_r = \sup\{\|\mathrm{Ind}_\mu(f)\| : \mu \text{ is a unit point mass on } E\}.$$

We denote by $C_0(G)$ the Banach space of continuous functions on G which vanish at infinity. Renault ([Ren80], Proposition II.4.2) showed that if G is an r-discrete groupoid with Haar system and σ a continuous 2-cocycle, then the injection $j : C_c(G) \to C_0(G)$,

extends to a norm-decreasing linear map $j : C_r^*(G, \sigma) \to C_0(G)$ which is one-to-one. Therefore, elements of $C_r^*(G, \sigma)$ can be viewed as continuous functions on G.

We will say that a locally compact groupoid with continuous Haar system (G, λ) is **amenable** if there exists a net $\{f_i\}$ in $C_c(G)$ such that:

Am1: the functions $u \mapsto \int |f_i(y)|^2 d\lambda^u(y)$ are uniformly bounded in the sup norm;

Am2: The functions $x \mapsto \int f_i(xy)\overline{f_i(y)}d\lambda^{d(x)}(y)$ converge to 1 uniformly on any compact subset of G.

The property of amenability of G guarantees that $C_r^*(G, \sigma)$ is nuclear and moreover that $C_r^*(G, \sigma)$ and $C^*(G, \sigma)$ coincide.

Suppose that a topological groupoid G is the union of an increasing sequence of open subgroupoids G_n all of which share the unit space of G. In that case we will say that G is the inductive limit of the sequence G_n. When G has a Haar system, we assume that its restriction endows each element of the sequence G_n with a Haar system. The inductive limit of amenable groupoids is amenable (see p.123 of [Ren80]).

Let G be a groupoid and let X be a set. A **groupoid action** of G on X (to the left) is given by a surjection $r : X \to G^{(0)}$ and a map $(\gamma, x) \mapsto \gamma \cdot x$ from $G * X := \{(\gamma, x) \mid d(\gamma) = r(x)\}$ to X, satisfying

(1) $r(x) \cdot x = x$ for all $x \in X$.
(2) $r(\gamma \cdot x) = r(\gamma)$ for every $(\gamma, x) \in G * X$.
(3) if $(\gamma_1, x) \in G * X$ and $(\gamma_2, \gamma_1) \in G^{(2)}$ then $(\gamma_2\gamma_1, x), (\gamma_2, \gamma_1 \cdot x) \in G * X$ and $\gamma_2 \cdot (\gamma_1 \cdot x) = (\gamma_2\gamma_1) \cdot x$.

We say that the (left) action is **free** when the equation $\gamma \cdot x = x$ implies that γ is a unit, namely $\gamma = d(\gamma) = r(x)$. In the case where the set X is itself a groupoid, we will say that the action is **transitive** if for every $x, y \in X$ such that $d(x) = d(y)$, there exists $\gamma \in G$ such that $x = \gamma \cdot y$. A groupoid homomorphism $\pi : X \to G$ is said to be **equivariant** with respect to the G-action if for every $(\gamma, x) \in G * X$ we have that $d(\gamma) = r(\pi(x))$ and $\pi(\gamma \cdot x) = \gamma\pi(x)$.

3. Bundles of Glimm Groupoids

Glimm groupoids are groupoids whose C^*-algebras are the well known Glimm (or UHF) C^*-algebras. They serve as basic examples in our study, however we present them here in a slightly non-standard fashion (compare with [Ren80]). Therefore we review some of their basic properties and we provide proofs for which we could not find a convenient reference in the literature.

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \dots\}$.

Definition 3.1. We shall say that a set Ω is **admissible** if there exists $S \subseteq \mathbb{N}$ of the form $S = \{0, 1, 2, \dots, n\}$ or $S = \mathbb{N}$ for which $\Omega \subseteq \prod_{k \in S} \mathbb{N}$ and furthermore $\Omega = \prod_{k \in S} \Omega_k$, where $\Omega_k \subseteq \mathbb{N}$ is finite and nonempty for every $k \in S$. We always endow Ω with the (compact) product topology of the discrete sets Ω_k, $k \in S$.

It will be useful to regard the elements of Ω as sequences $\alpha = (\alpha_0, \alpha_1, \dots)$ or $(n+1)$-tuples $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$.

We define an equivalence relation on an admissible set Ω as follows. For $\alpha, \beta \in \Omega$, we say that

$$\alpha \sim \beta \Leftrightarrow \alpha_k = \beta_k \text{ except for finitely many values of } k.$$

Notice that in the case when Ω is finite any two points are equivalent. The aforementioned equivalence relation gives rise to a principal groupoid

$$\mathcal{R}(\Omega) = \{(\alpha, \beta) \in \Omega \times \Omega : \alpha \sim \beta\}$$

where the pair (α, β), (γ, δ) is composable if and only if $\beta = \gamma$, and $(\alpha, \beta)^{-1} = (\beta, \alpha)$.

We define a topology on $\mathcal{R}(\Omega)$ as follows. For every $(\alpha, \beta) \in \mathcal{R}(\Omega)$ and $N \in \mathbb{N}$ such that $\alpha_n = \beta_n$ for $n > N$, let

$$\mathcal{O}_{(\alpha,\beta)}(N) = \{(\gamma, \delta) \in \mathcal{R}(\Omega) : \forall k \leq N, \gamma_k = \alpha_k, \delta_k = \beta_k \text{ and } \forall k > N, \gamma_k = \delta_k\}.$$

The collection $\{\mathcal{O}_\xi(N) : \xi \in \mathcal{R}(\Omega), N \in \mathbb{N}\}$ is a basis for the topology of $\mathcal{R}(\Omega)$. In terms of convergence of nets, a net $(\alpha_\lambda, \beta_\lambda)$ in $\mathcal{R}(\Omega)$ converges to a point $(\alpha, \beta) \in \mathcal{R}(\Omega)$ if and only if for every $n \in \mathbb{N}$ such that $\alpha_k = \beta_k$ for $k > n$ there exists λ_0 such that for $\lambda \geq \lambda_0$, $(\alpha_\lambda)_k = \alpha_k$ and $(\beta_\lambda)_k = \beta_k$ when $k \leq n$ and $(\alpha_\lambda)_k = (\beta_\lambda)_k$ when $k > n$.

Note that the topology of $\mathcal{R}(\Omega)$ has a countable basis since $\mathcal{O}_{(\alpha,\beta)}(N) = \mathcal{O}_{(\gamma,\delta)}(N)$ whenever $\alpha_k = \gamma_k$ and $\beta_k = \delta_k$ for $k \leq N$.

Definition 3.2. The principal topological groupoid $\mathcal{R}(\Omega)$ will be called the **Glimm groupoid of an admissible set Ω**.

For example, when Ω is finite, $\mathcal{R}(\Omega)$ is the groupoid $\Omega \times \Omega$ corresponding to the trivial equivalence relation on Ω, and in particular it is compact. When Ω is infinite this need not be the case. Nevertheless, its unit space is still compact since the map $\alpha \mapsto (\alpha, \alpha)$ implements a homeomorphism $\Omega \simeq \mathcal{R}(\Omega)^{(0)}$.

Proposition 3.3. *The Glimm groupoid of an admissible set Ω is a locally compact Hausdorff groupoid which is principal and étale. Furthermore, the C^*-algebra $C^*(\mathcal{R}(\Omega))$ is a Glimm algebra (also called uniformly hyperfinite or UHF).*

Proof. This is a straightforward verification, except for the statement regarding the nature of the C^*-algebra, which follows from Renault's study of Glimm groupoids and AF-groupoids in [Ren80] (see p.128 and results thereafter). $\square$

Let $\mathcal{T}$ be a second countable compact Hausdorff space.

Definition 3.4. The **bundle of Glimm groupoids of an admissible set Ω over $\mathcal{T}$**, which will be denoted by $\mathcal{T} * \mathcal{R}(\Omega)$, is the set $\mathcal{T} \times \mathcal{R}(\Omega)$ endowed with the pointwise groupoid structure and the product topology. It will be convenient to work with $\mathcal{T} * \mathcal{R}(\Omega)$ in the following presentation:

$$\mathcal{T} * \mathcal{R}(\Omega) = \{(\alpha, t, \beta) : t \in \mathcal{T}, (\alpha, \beta) \in \mathcal{R}(\Omega)\}$$

In this notation a pair (α, t, β), (γ, s, δ) is composable if and only if $\beta = \gamma$ and $t = s$, in which case their product is (α, t, δ), and $(\alpha, t, \beta)^{-1} = (\beta, t, \alpha)$.

The groupoid $\mathcal{T} * \mathcal{R}(\Omega)$ arises naturally from the equivalence relation on $\mathcal{T} \times \Omega$ which is defined by $(t, \alpha) \sim (s, \beta)$ if and only if $t = s$ and $\alpha \sim \beta$. In particular, its unit space is given by $\{(\alpha, t, \alpha) : t \in \mathcal{T}, \alpha \in \Omega\} = \mathcal{T} * \mathcal{R}^{(0)}(\Omega)$.

We collect many of the properties of $\mathcal{T} * \mathcal{R}(\Omega)$ in the following proposition, although we omit its proof since it is a routine verification. Recall that locally compact second countable Hausdorff spaces are metrizable by Urysohn's metrization theorem (see [Kel75] pp. 125 and 147).

Proposition 3.5. *Given an admissible set Ω and a compact second countable Hausdorff space $\mathcal{T}$, the groupoid $\mathcal{T} * \mathcal{R}(\Omega)$ is locally compact, second countable, Hausdorff, metrizable, principal and étale. Furthermore, its unit space is compact since it is naturally homeomorphic to $\mathcal{T} \times \Omega$.*

We now address the issue of amenability. When Ω is finite, as we have remarked, the groupoid $\mathcal{R}(\Omega)$ is compact. Therefore the étale groupoid $\mathcal{T} * \mathcal{R}(\Omega)$ is compact, and it follows that it is amenable.

We now turn to the case when Ω is infinite and $\Omega = \prod_{k\in\mathbb{N}} \Omega_k$.

We introduce a bit more notation. For every $n \in \mathbb{N}$, we will denote $\Omega^{(n)} = \prod_{k=0}^{n} \Omega_k$. Note that $\Omega^{(n)}$ is also an admissible set. Given $\alpha \in \Omega$ and $n \in \mathbb{N}$, we shall denote $\alpha|n = (\alpha_0, \alpha_1, \ldots, \alpha_n)$. Notice that, using this notation, we have $\Omega^{(n)} = \{\alpha|n : \alpha \in \Omega\}$.

Definition 3.6. For each $n \geq 0$, denote by $\mathfrak{X}_n$ the subset of $\mathcal{T}*\mathcal{R}(\Omega)$ given by

$$\mathfrak{X}_n = \{(\alpha, t, \beta) \in \mathcal{T}*\mathcal{R}(\Omega) : \alpha_k = \beta_k \text{ for } k > n\}.$$

Notice that the sets $\mathfrak{X}_n$ are nested: $\mathfrak{X}_n \subseteq \mathfrak{X}_{n+1}$, and that $\mathcal{T}*\mathcal{R}(\Omega) = \bigcup_{n\in\mathbb{N}} \mathfrak{X}_n$.

Remark 3.7. *We can restate the convergence in $\mathcal{T}*\mathcal{R}(\Omega)$ as follows. We will denote by $p_\mathcal{T} : \mathcal{T}*\mathcal{R}(\Omega) \to \mathcal{T}$ and $p_n : \mathcal{T}*\mathcal{R}(\Omega) \to \mathcal{R}(\Omega^{(n)})$ the canonical projections given by $p_\mathcal{T}(\alpha, t, \beta) = t$ and $p_n(\alpha, t, \beta) = (\alpha|n, \beta|n)$. A net ξ_λ converges to ξ in $\mathcal{T}*\mathcal{R}(\Omega)$ if and only if*

(1) *$p_\mathcal{T}(\xi_\lambda) \to p_\mathcal{T}(\xi)$.*
(2) *$\forall n \in \mathbb{N}$ such that $\xi \in \mathfrak{X}_n$, there exists λ_0 such that for any $\lambda \geq \lambda_0$, $\xi_\lambda \in \mathfrak{X}_n$ and $p_n(\xi_\lambda) = p_n(\xi)$.*

We also note that the projections $p_\mathcal{T}$ and p_n are continuous.

Lemma 3.8. *For every $n \geq 0$ the set $\mathfrak{X}_n$ is a compact open amenable étale subgroupoid of $\mathcal{T}*\mathcal{R}(\Omega)$ such that $\mathfrak{X}_n^{(0)} = [\mathcal{T}*\mathcal{R}(\Omega)]^{(0)}$.*

Proof. It is clear that $\mathfrak{X}_n$ is a subgroupoid of $\mathcal{T}*\mathcal{R}(\Omega)$, and $\mathfrak{X}_n$ contains $[\mathcal{T}*\mathcal{R}(\Omega)]^{(0)}$, hence $\mathfrak{X}_n^{(0)} = [\mathcal{T}*\mathcal{R}(\Omega)]^{(0)}$. It follows immediately from Remark 3.7 (2) that $\mathfrak{X}_n$ is open. In order to show that $\mathfrak{X}_n$ is closed, suppose that $\xi_\lambda \to \xi$ and $\xi_\lambda \in \mathfrak{X}_n$ for all λ. Since $\xi \in \mathcal{T}*\mathcal{R}(\Omega) = \cup_{n\in\mathbb{N}}\mathfrak{X}_n, \exists N \in \mathbb{N}$ such that $\xi \in \mathfrak{X}_N$. If $N \leq n$ we have that $\xi \in \mathfrak{X}_N \subseteq \mathfrak{X}_n$ and in this case we are done. Now assume $N > n$. There exists λ_0 such that for $\lambda \geq \lambda_0$, $\xi_\lambda \in \mathfrak{X}_N$ and $p_N(\xi_\lambda) = p_N(\xi)$. Thus, if $\xi_\lambda = (\alpha_\lambda, t_\lambda, \beta_\lambda)$ and $\xi = (\alpha, t, \beta)$, we have for $\lambda \geq \lambda_0$ that $\alpha_k = (\alpha_\lambda)_k = (\beta_\lambda)_k = \beta_k$ for $k = n + 1, \ldots, N$. Hence, $\xi \in \mathfrak{X}_n$.

Let $Q_n = \Omega^{(n)} \times \Omega^{(n)} \times \prod_{k=n+1}^{\infty} \Omega_k$ endowed with the product topology, with respect to which it is compact. Let $j : Q_n \to \mathcal{R}(\Omega)$ be the map given by $j(x, y, \omega) = (x\omega, y\omega)$, where $x\omega$ denotes the concatenation of x and ω. This map is clearly injective and it is continuous, therefore $j(Q_n)$ is compact in $\mathcal{R}(\Omega)$. Now observe that $\mathfrak{X}_n \subseteq \mathcal{T} \times j(Q_n)$, hence it is a closed subset of a compact set of $\mathcal{T}*\mathcal{R}(\Omega)$ and we conclude that it is compact.

The groupoid $\mathfrak{X}_n$ is étale because it is an open subgroupoid of an étale groupoid. Since $r : \mathcal{T}*\mathcal{R}(\Omega) \to [\mathcal{T}*\mathcal{R}(\Omega)]^{(0)}$ is a local homeomorphism, its restriction to the open set $\mathfrak{X}_n$ is still a local homeomorphism.

Finally, $\mathfrak{X}_n$ is amenable because it is compact. $\qquad\square$

In summary, we have shown that $\mathcal{T}*\mathcal{R}(\Omega)$ is an inductive limit of compact open amenable subgroupoids, and it follows in particular that it is also amenable. We have proven the following result.

Proposition 3.9. *Given an admissible set Ω and a compact second countable Hausdorff space $\mathcal{T}$, the groupoid $\mathcal{T}*\mathcal{R}(\Omega)$ is amenable.*

Observe that when $\mathcal{T} = \{\mathrm{pt}\}$, we have that $\mathcal{T}*\mathcal{R}(\Omega) \simeq \mathcal{R}(\Omega)$. In particular, the latter is amenable.

Although we chose an alternative path, it is possible to define the topology of $\mathcal{T}*\mathcal{R}(\Omega)$ in terms of the inductive limit of the sequence $\mathfrak{X}_n$ (see [Ren80], p. 122).

4. Groupoids for Ordered Cover refinements

Definition 4.1. Let $\mathcal{U}$ and $\mathcal{W}$ be open covers of $\mathcal{T}$. We say that $\mathcal{W}$ is an **intersection refinement** of $\mathcal{U}$ and denote it $\mathcal{U} \leq \mathcal{W}$ if there exists an open cover $\mathcal{V}$ such that $\mathcal{W} = \mathcal{U} \cap \mathcal{V} = \{U_i \cap V_j \mid U_i \in \mathcal{U}, V_j \in \mathcal{V}\}$.

It is easy to see that the set of intersection refinements is cofinal in the set of all refinements.

Given any family of open covers $V = \{V^{(0)}, V^{(1)}, V^{(2)}...\}$, we obtain a sequence of intersection refinements $W^{(0)} \leq W^{(1)} \leq W^{(2)} \leq ...$ by defining $W^{(n)} = V^{(0)} \cap V^{(1)} \cap V^{(2)}... \cap V^{(n)}$. We present this exact construction using the notation and terminology we introduced in the previous section.

For each $k \in \mathbb{N}$, let $V^{(k)} = \{V_0^{(k)}, V_1^{(k)}, \ldots V_{n_k}^{(k)}\}$ be an open cover for $\mathcal{T}$. We emphasize that we allow repetitions, i.e. $V_i^{(k)} = V_j^{(k)}$ for $i \neq j$. The family $V = \{V^{(0)}, V^{(1)}, V^{(2)}...\}$ is then a family of open covers of $\mathcal{T}$. We can now apply the terminology of the previous section. For each $k \in \mathbb{N}$, denote $\Omega_k = \{0, 1, \ldots, n_k\}$ and let $\Omega = \prod_{k=0}^{\infty} \Omega_k$ be the **admissible set corresponding to** V. The set Ω is in fact admissible in the sense of Definition 3.1, as are the sets $\Omega^{(n)} = \prod_{k=0}^{n} \Omega_k$, for every $n \in \mathbb{N}$.

If $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega^{(n)}$, we set:

$$W_\alpha = \bigcap_{k=0}^{n} V_{\alpha_k}^{(k)}.$$

Note that W_α is an open (possibly empty) set for every α, being a finite intersection of open sets. Moreover, for every $n \in \mathbb{N}$, $W^{(n)} = \{W_\alpha : \alpha \in \Omega^{(n)}\}$ is an open cover of $\mathcal{T}$, and $W^{(0)} \leq W^{(1)} \leq W^{(2)} \leq ...$ is a sequence of intersection refinements corresponding to V. In light of this, we will call V an **ordered cover refinement**.

When $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots) \in \Omega$, the infinite intersection $W_\alpha = \bigcap_{k=0}^{\infty} V_{\alpha_k}^{(k)}$ is only a G_δ set. Thus, in this case $\{W_\alpha : \alpha \in \Omega\}$ is not an open cover of $\mathcal{T}$.

Definition 4.2. For every $N \in \mathbb{N}$, we define the groupoid

$$G_N(V) = \{(\alpha, t, \beta) : \alpha, \beta \in \Omega^{(N)}, t \in W_\alpha \cap W_\beta\}$$

where a pair (α, t, β), (γ, s, δ) is composable if and only if $\beta = \gamma$ and $t = s$, in which case their product is (α, t, δ), and $(\alpha, t, \beta)^{-1} = (\beta, t, \alpha)$. The topology of $G_N(V)$ is the relative topology from its natural inclusion into $\mathcal{T} * \mathcal{R}(\Omega^{(N)})$.

The topology of $G_N(V)$ has as basis the collection of all sets

$$Z_{\alpha,\beta,U} = \{(\alpha, t, \beta) \in G_N(V) : t \in U\}$$

where $(\alpha, \beta) \in \mathcal{R}(\Omega^{(N)})$ and $U \subseteq \mathcal{T}$ is open. The sets $Z_{\alpha,\beta,U}$ implicitly depend on N.

For every $N \in \mathbb{N}$ the groupoid $G_N(V)$ arises naturally from the restriction of the equivalence relation introduced on $\mathcal{T} \times \Omega^{(N)}$ to the set $\{(t, \alpha) \in \mathcal{T} \times \Omega^{(N)} : t \in W_\alpha\}$.

The groupoid $G_N(V)$ is precisely the groupoid $\mathcal{G}(W^{(N)})$ corresponding to the Raeburn-Taylor C^*-algebra of the finite open cover $W^{(N)}$ of $\mathcal{T}$. Presenting the elements in the form (α, t, β), we keep track of all the covers $V^{(0)}, V^{(1)}, V^{(2)}, ..., V^{(N)}$ from which $W^{(N)}$ was obtained. The groupoid $G_N(V)$ has the following basic properties.

Proposition 4.3 ([RT85])**.** *For every $N \in \mathbb{N}$ the groupoid $G_N(V)$ is locally compact, Hausdorff, principal and étale.*

Definition 4.4. For every $N \in \mathbb{N}$, define the groupoid

$$\check{G}_N(V) = \{(\alpha, t, \beta) : (\alpha, \beta) \in \mathcal{R}(\Omega^{(N)}), \ t \in \overline{W_\alpha} \cap \overline{W_\beta}\}$$

with the following operations: a pair (α, t, β), (γ, s, δ) is composable if and only if $\beta = \gamma$ and $t = s$, in which case their product is (α, t, δ), and $(\alpha, t, \beta)^{-1} = (\beta, t, \alpha)$. The topology of $\check{G}_N(V)$ is the relative topology from its natural inclusion into $\mathcal{T} * \mathcal{R}(\Omega^{(N)})$.

The sets of the form

$$\check{Z}_{\alpha,\beta,U} = \{(\alpha, t, \beta) \in \check{G}_N(V) : t \in U\}$$

where $(\alpha, \beta) \in \mathcal{R}(\Omega^{(N)})$ and $U \subseteq \mathcal{T}$ is open, constitute a basis for the topology of $\check{G}_N(V)$.

Proposition 4.5. *For every $N \in \mathbb{N}$, $\check{G}_N(V)$ is a compact Hausdorff r-discrete principal groupoid. The embedding $J : G_N(V) \to \check{G}_N(V)$ is a continuous open mapping and a groupoid homomorphism.*

Proof. The groupoid $\check{G}_N(V)$ arises from the restriction of the equivalence relation of $\mathcal{T} \times \Omega$ to the set $\{(t, \alpha) \in \mathcal{T} \times \Omega^{(N)} : t \in \overline{W_\alpha}\}$, hence it is a principal groupoid. We claim that $\check{G}_N(V)$ is a closed subset of the compact Hausdorff space $\mathcal{T} * \mathcal{R}(\Omega^{(N)})$ $\simeq \mathcal{T} \times \Omega^{(N)} \times \Omega^{(N)}$, hence compact and Hausdorff in the relative topology. Let $(\alpha_\lambda, t_\lambda, \beta_\lambda)$ be a net in $\check{G}_N(V)$ converging to $(\alpha, t, \beta) \in \mathcal{T} * \mathcal{R}(\Omega^{(N)})$. Then there exists λ_0 such that for $\lambda \geq \lambda_0$, $\alpha_\lambda = \alpha$ and $\beta_\lambda = \beta$, hence $t_\lambda \in \overline{W_\alpha} \cap \overline{W_\beta}$. Since $t_\lambda \to t$, it follows that $t \in \overline{W_\alpha} \cap \overline{W_\beta}$ and $(\alpha, t, \beta) \in \check{G}_N(V)$. Finally, the unit space $\check{G}_N^{(0)}(V)$ of $\check{G}_N(V)$ is open, since $\check{G}_N^{(0)}(V) = \bigcup_{\alpha \in \Omega^{(N)}} \check{Z}_{\alpha,\alpha,\mathcal{T}}$. Hence $\check{G}_N(V)$ is r-discrete.

It is clear that the mapping J is a continuous groupoid homomorphism. In order to prove that it is an open mapping, let $W \subseteq G_N(V)$ be open and fix $(\alpha, t, \beta) \in W$. Now let $U \subseteq W_\alpha \cap W_\beta$ be an open neighborhood of t in $\mathcal{T}$ such that $Z_{\alpha,\beta,U} \subseteq W$. We have that if $(\gamma, s, \delta) \in \check{Z}_{\alpha,\beta,U}$, then $\gamma = \alpha$, $\delta = \beta$ and $s \in U \subseteq W_\alpha \cap W_\beta$, hence $(\gamma, s, \delta) \in G_N(V)$. In other words, $\check{Z}_{\alpha,\beta,U} \subseteq G_N(V)$. Since it follows from the definitions that $Z_{\alpha,\beta,U} = \check{Z}_{\alpha,\beta,U} \cap G_N(V)$, we have that in this case $Z_{\alpha,\beta,U} = \check{Z}_{\alpha,\beta,U}$, and therefore $\check{Z}_{\alpha,\beta,U} \subseteq W$. It follows that W is a union of open sets of $\check{G}_N(V)$, thus open. $\qquad\square$

Our goal is to "take the limit as N goes to ∞" of the Raeburn-Taylor groupoids $G_N(V)$. We are aiming for a groupoid $G(V)$ which is locally compact, Hausdorff, principal and étale. A first step in this direction is to consider the following groupoid:

Definition 4.6. We shall refer to the topological groupoid

$$G_\infty(V) = \{(\alpha, t, \beta) \mid (\alpha, \beta) \in \mathcal{R}(\Omega), \ t \in W_\alpha \cap W_\beta\}$$

endowed with the following structure. A pair (α, t, β), (γ, s, δ) is composable if and only if $\beta = \gamma$ and $t = s$, in which case their product is (α, t, δ). The inverse is given by $(\alpha, t, \beta)^{-1} = (\beta, t, \alpha)$. The topology of $G_\infty(V)$ is the relative topology from its natural inclusion in $\mathcal{T} * \mathcal{R}(\Omega)$.

It follows from the inclusion of $G_\infty(V)$ into $\mathcal{T} * \mathcal{R}(\Omega)$ that $G_\infty(V)$ is in fact a groupoid. But in general, this natural candidate for $G(V)$ fails to be locally compact (see Example 5.7). Furthermore, the topological closure of $G_\infty(V)$ in $\mathcal{T} * \mathcal{R}(\Omega)$ need not be closed under the multiplication induced from $\mathcal{T} * \mathcal{R}(\Omega)$ (see Example 4.12), and thus it is not a groupoid. We are led to consider $\check{G}_\infty(V)$ which is the algebraic closure of $\overline{G_\infty(V)}$ inside $\mathcal{T} * \mathcal{R}(\Omega)$. Although $\check{G}_\infty(V)$ is a locally compact groupoid, it is also lacking because it need not be étale (see Example 5.6). Ultimately we identify a groupoid $G(V)$ which is a subgroupoid of $\check{G}_\infty(V)$ containing $G_\infty(V)$ and which satisfies all the above desired properties. Having outlined our plan, we now provide the definitions of $\check{G}_\infty(V)$ and $G(V)$, establish their properties, and give some examples.

Definition 4.7. We endow the set

$$\check{G}_\infty(V) = \{(\alpha, t, \beta) : (\alpha, \beta) \in \mathcal{R}(\Omega),\ t \in (\bigcap_{N=0}^\infty \overline{W_{\alpha|N}}) \cap (\bigcap_{N=0}^\infty \overline{W_{\beta|N}})\}$$

with the following groupoid operations: a pair (α, t, β), (γ, s, δ) is composable if and only if $\beta = \gamma$ and $t = s$, in which case their product is (α, t, δ). The inverse is given by $(\alpha, t, \beta)^{-1} = (\beta, t, \alpha)$. The topology of $\check{G}_\infty(V)$ is the relative topology from its natural inclusion into $\mathcal{T} * \mathcal{R}(\Omega)$. As we will see shortly, $\check{G}_\infty(V)$ is a groupoid, and it is the algebraic closure of $\overline{G_\infty(V)}$.

Definition 4.8. We endow the set

$$G(V) = \{(\alpha, t, \beta) \in \check{G}_\infty(V) : \exists n \in \mathbb{N}, (\alpha|n, t, \beta|n) \in G_n(V) \text{ and } \alpha_k = \beta_k \text{ for } k > n\}$$

with the groupoid operations and the relative topology inherited from its natural inclusion into $\check{G}_\infty(V)$. We will soon verify that $G(V)$ is in fact a subgroupoid of $\check{G}_\infty(V)$.

In order to lighten the notation, henceforth we will fix V, the sequence of intersection refinements, and we will refrain from marking the dependence of the groupoids on the sequence. For example, we will denote $\check{G}_\infty(V)$ by $\check{G}_\infty$.

Note that for any $N \in \mathbb{N}$ and $\alpha \in \Omega^{(N)}$ we have that $\bigcap_{n=0}^N \overline{W_{\alpha|n}} = \overline{W_\alpha}$. In light of this we view $\check{G}_\infty$ as the counterpart of $\check{G}_N$ when replacing N with ∞.

It may not be immediately apparent that G is closed under the operations of $\check{G}_\infty$. We address this issue in the following proposition.

Proposition 4.9. *$\check{G}_\infty$ is a principal groupoid containing G_∞ and G as subgroupoids. In particular, G is a principal groupoid.*

Proof. The operations on $\check{G}_\infty$ correspond to the equivalence relation on $\mathcal{T} \times \Omega$ corresponding to $\mathcal{T} * \mathcal{R}(\Omega)$ but restricted to the set $\{(t, \alpha) \in \mathcal{T} \times \Omega \mid t \in \bigcap_{N=0}^\infty \overline{W_{\alpha|N}}\}$. Therefore $\check{G}_\infty$ is a principal groupoid. It is straightforward to check that G_∞ is a subgroupoid of $\check{G}_\infty$. Now suppose that (α, t, β) and (δ, s, γ) are elements of G which are composable in $\check{G}_\infty$. Then we have that $t = s$, $\beta = \delta$, and there are $n, m \in \mathbb{N}$ such that $t \in W_{\alpha|n} \cap W_{\beta|n}$, $t \in W_{\beta|m} \cap W_{\gamma|m}$, $\alpha_k = \beta_k$ for $k > n$ and $\beta_j = \gamma_j$ for $j > m$. Let us assume without loss of generality that $n \leq m$. We can then write:

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots \alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots)$$
$$\beta = (\beta_0, \beta_1, \ldots, \beta_n, \alpha_{n+1}, \ldots \alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots)$$
$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_m, \alpha_{m+1}, \alpha_{m+2}, \ldots)$$

Notice that since $t \in W_{\beta|m}$ and $t \in W_{\alpha|n}$, we actually have that $t \in W_{\alpha|m}$. Since $t \in W_{\gamma|m}$, we conclude that $(\alpha, t, \gamma) \in G$. It is clear that G is closed under inverses, therefore it is a subgroupoid of $\check{G}_\infty$.

Finally, G_∞ and G are principal groupoids because they are subgroupoids of a principal groupoid. □

The following remark will prove useful.

Remark 4.10. *Suppose O is an open set, A is a set and $x \in O \cap \overline{A}$. Then $x \in \overline{O \cap A}$.*

The proof is easy. Given a net x_λ in A converging to x, there exists λ_0 such that for $\lambda \geq \lambda_0$ we have $x_\lambda \in O$. Thus for $\lambda \geq \lambda_0$ we have $x_\lambda \in O \cap A$ and $x_\lambda \to x$.

Proposition 4.11. *The topological closure of G_∞ in $\mathcal{T} * \mathcal{R}(\Omega)$ is given by*

(4.11.1) $$\overline{G_\infty} = \{(\alpha, t, \beta) \in \mathcal{T} * \mathcal{R}(\Omega) : t \in \cap_N \overline{W_{\alpha|N} \cap W_{\beta|N}}\}.$$

Moreover, The groupoid $\check{G}_\infty$ is the algebraic closure of $\overline{G_\infty}$ in $\mathcal{T} * \mathcal{R}(\Omega)$.

Proof. Suppose $(\alpha, t, \beta) \in \mathcal{T} * \mathcal{R}(\Omega)$ is the limit of a net $(\alpha_\lambda, t_\lambda, \beta_\lambda)_{\lambda \in \Lambda}$ in G_∞. Fix $N \in \mathbb{N}$. There exists $\lambda_0 \in \Lambda$ such that for $\lambda \geq \lambda_0$, $\alpha_\lambda|N = \alpha|N$, $\beta_\lambda|N = \beta|N$. In particular, for $\lambda \geq \lambda_0$, $t_\lambda \in W_{\alpha_\lambda|N} \cap W_{\beta_\lambda|N} = W_{\alpha|N} \cap W_{\beta|N}$. Since $t_\lambda \to t$ we have that $t \in \overline{W_{\alpha|N} \cap W_{\beta|N}}$.

Conversely, suppose $(\alpha, t, \beta) \in \mathcal{T} * \mathcal{R}(\Omega)$ and $t \in \cap_N \overline{W_{\alpha|N} \cap W_{\beta|N}}$. Let S be a local basis of open neighborhoods of t, partially ordered by reverse inclusion, and let $\Lambda = \mathbb{N} \times S$ be the directed set with the product partial ordering. Let $\lambda = (N, A)$ be fixed. Notice that $t \in \overline{W_{\alpha|N} \cap W_{\beta|N}}$, hence we can choose $t_\lambda \in A \cap W_{\alpha|N} \cap W_{\beta|N}$. Define $\alpha_\lambda|N = \alpha|N$, $\beta_\lambda|N = \beta|N$, and for $n > N$ set $(\alpha_\lambda)_n = (\beta_\lambda)_n$ to be any value so that $t_\lambda \in V^{(n)}_{(\alpha_\lambda)_n}$. This is possible because V is an ordered cover refinement. Now it is clear that $(\alpha_\lambda, t_\lambda, \beta_\lambda)$ is a net in G_∞ which converges to (α, t, β).

Finally, we show that $\check{G}_\infty$ is the algebraic closure of $\overline{G_\infty}$ in $\mathcal{T} * \mathcal{R}(\Omega)$. It follows from the definition of $\check{G}_\infty$ in conjunction with (4.11.1) that $\overline{G_\infty} \subseteq \check{G}_\infty$. Therefore, in order to prove the statement, it suffices to show that for every $z \in \check{G}_\infty$ there exist $x, y \in \overline{G_\infty}$ such that $z = xy$.

Fix $z = (\alpha, t, \beta) \in \check{G}_\infty$. There exists $M \in \mathbb{N}$ such that $\alpha_n = \beta_n$ for $n > M$. Now pick any $\gamma \in \Omega$ such that $t \in W_{\gamma|M}$ (again this is possible since V is an ordered cover refinement) and $\gamma_n = \alpha_n$ for $n > M$. Then set $x = (\alpha, t, \gamma)$ and $y = (\gamma, t, \beta)$.

We claim that $x, y \in \overline{G_\infty}$. We prove it for x, since an analogous argument will yield the result for y. We must prove that $t \in \overline{W_{\alpha|N} \cap W_{\gamma|N}}$ for all $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$, and recall that $t \in \overline{W_{\alpha|N}}$. Suppose first that $N \leq M$. Then $W_{\gamma|N}$ is open and contains t, hence by Remark 4.10 we have $t \in \overline{W_{\alpha|N} \cap W_{\gamma|N}}$. When $N > M$ notice that $\overline{W_{\alpha|N} \cap W_{\gamma|N}} = \overline{W_{\alpha|N} \cap W_{\gamma|M}}$ and the latter contains t by Remark 4.10. Finally, it is clear that $z = xy$. $\square$

Example 4.12 (*In general $G_\infty \neq \overline{G_\infty}$ and $\overline{G_\infty} \neq \check{G}_\infty$*). Consider the ordered cover refinement V given as follows: for every k let $V^{(k)}_0 = [-1, 0)$, $V^{(k)}_1 = (0, 1]$ and $V^{(k)}_2 = [-1, 1]$, and consider the following elements (we denote the infinite repetition of a number by placing a bar over it):

$$g = (0000\overline{0}, 0, 0000\overline{0}) \qquad\qquad x = (0222\overline{2}, 0, 2222\overline{2})$$
$$y = (2222\overline{2}, 0, 1222\overline{2}) \qquad\qquad z = (0222\overline{2}, 0, 1222\overline{2})$$

We have immediately that $G_\infty \neq \overline{G_\infty}$, since we have a net $(000\overline{0}, -1/n, 0000\overline{0})$ in G_∞ converging to $g \notin G_\infty$.

Observe that $z \in \check{G}_\infty$ but $z \notin \overline{G_\infty}$ by (4.11.1) since $\overline{W_0 \cap W_1} = \emptyset$. Incidentally, this shows already that $\overline{G_\infty}$ is not closed under multiplication, for in that case it would be equal to its algebraic closure $\check{G}_\infty$ since it is obviously closed under inverses. More concretely, notice that $x, y \in \overline{G_\infty}$ and $z = xy$, but $z \notin \overline{G_\infty}$.

We remark that in general, one can verify that $\check{G}_\infty = \overline{G_\infty} \cdot \overline{G_\infty}$.

We have seen that if G is a topological groupoid and S is a subset closed under the groupoid operations of G, in general $\overline{S}$ need not be closed under the operations of G. This is different from the category of groups, where the closure of a subgroup is automatically a subgroup. We point out that from the axioms in the definition of a groupoid it follows that $(\overline{S} \times \overline{S}) \cap G^{(2)}$ determines the set of composable pairs for $\overline{S}$. In particular, $\overline{G_\infty}$ cannot be made into a groupoid with the restriction of the operations of $\mathcal{T} * \mathcal{R}(\Omega)$ by attempting to declare a smaller set of composable pairs.

5. PROPERTIES OF $\check{G}_\infty$ AND G

Proposition 5.1. *$\check{G}_\infty$ is a closed subset of $\mathcal{T} * \mathcal{R}(\Omega)$, therefore it is a metrizable locally compact groupoid.*

Proof. The topological space $\check{G}_\infty$ is metrizable because it is a subspace of $\mathcal{T}*\mathcal{R}(\Omega)$ which is metrizable, by Proposition 3.5.

Let $x_i = (\alpha_i, t_i, \beta_i)$ be a sequence in $\check{G}_\infty$ converging to $x = (\alpha, t, \beta)$ in $\mathcal{T}*\mathcal{R}(\Omega)$, that is to say, $t_i \to t$ in $\mathcal{T}$ and $(\alpha_i, \beta_i) \to (\alpha, \beta)$ in $\mathcal{R}(\Omega)$. In order to show that $x \in \check{G}_\infty$, we need only to show that $t \in (\cap_{n\in\mathbb{N}}\overline{W_{\alpha|n}}) \cap (\cap_{n\in\mathbb{N}}\overline{W_{\beta|n}})$. For each $k > 0$, denote $F_k = (\cap_{n=1}^{k}\overline{W_{\alpha|n}}) \cap (\cap_{n=1}^{k}\overline{W_{\beta|n}})$; this is a sequence of decreasing sets and of course we want to show that $t \in \cap_{k\in\mathbb{N}}F_k$. Since $(\alpha_i, \beta_i) \to (\alpha, \beta)$ in $\mathcal{R}(\Omega)$, for every $k > 0$ there exists $i_k > 0$ such that for $i \geq i_k$, $(\alpha_i)_n = \alpha_n$ and $(\beta_i)_n = \beta_n$ for $n \leq k$. Since $x_{i_k} \in \check{G}_\infty$, we have that $t_{i_k} \in (\cap_{n\in\mathbb{N}}\overline{W_{\alpha_{i_k}|n}}) \cap (\cap_{n\in\mathbb{N}}\overline{W_{\beta_{i_k}|n}})$, therefore $t_{i_k} \in (\cap_{n=1}^{k}\overline{W_{\alpha|n}}) \cap (\cap_{n=1}^{k}\overline{W_{\beta|n}}) = F_k$. Since $\{F_k : k > 0\}$ is a decreasing sequence of closed sets and $t_{i_k} \to t$, we have that $t \in F_k$ for every k. Therefore $x \in \check{G}_\infty$ and we have shown that $\check{G}_\infty$ is a closed subset of $\mathcal{T}*\mathcal{R}(\Omega)$. In particular, as a closed subset of a locally compact set, $\check{G}_\infty$ is locally compact in its relative topology. $\square$

The following subsets of $\check{G}_\infty$ will play an important role in the sequel.

Definition 5.2. For each $n \geq 0$, define the subset of $\check{G}_\infty$ given by

$$\mathcal{Y}_n = \check{G}_\infty \cap \mathfrak{X}_n = \{(\alpha, t, \beta) \in \check{G}_\infty : \alpha_k = \beta_k \text{ for } k > n\}.$$

Notice that the sets $\mathcal{Y}_n$ are nested: $\mathcal{Y}_n \subseteq \mathcal{Y}_{n+1}$, and that $\check{G}_\infty = \bigcup_{n\in\mathbb{N}}\mathcal{Y}_n$. Furthermore, the convergence in $\check{G}_\infty$ can be stated using the sets $\mathcal{Y}_n$ instead of $\mathfrak{X}_n$ in Remark 3.7.

Remark 5.3. *For every $n \geq 0$ the set $\mathcal{Y}_n$ is clearly a subgroupoid of $\check{G}_\infty$. Furthermore it is a compact open subset of $\check{G}_\infty$. This follows immediately from Lemma 3.8 and Proposition 5.1, since $\mathcal{Y}_n = \check{G}_\infty \cap \mathfrak{X}_n$. We also point out that $\mathcal{Y}_n^{(0)} = \check{G}_\infty^{(0)}$ since $\mathcal{Y}_n$ contains $\check{G}_\infty^{(0)}$.*

Definition 5.4. For any open subset U of $\mathcal{T}$, $n \in \mathbb{N}$ and $(\alpha, \beta) \in \mathcal{R}(\Omega^{(n)})$, define

$$\check{Z}_{\alpha,\beta,U,n} = \{(\gamma, t, \delta) \in \check{G}_\infty : t \in U, \gamma|n = \alpha, \delta|n = \beta, \text{ and } \forall k > n, \gamma_k = \delta_k\}$$

It is easy to check that the collection of all such sets forms a basis for the topology of $\check{G}_\infty$. It is useful to notice that this is in fact a basis of open G-sets, i.e. sets where r and d are injective. Therefore we also have that G_∞ and G each have a basis of open G-sets in the relative topology.

Theorem 5.5. *The groupoid $\check{G}_\infty$ is locally compact, second countable, Hausdorff, metrizable, principal and r-discrete.*

Proof. We have already shown that $\check{G}_\infty$ is principal in Proposition 4.9, and that it is metrizable and locally compact in Proposition 5.1. It also inherits from $\mathcal{T}*\mathcal{R}(\Omega)$ the properties of being second countable and Hausdorff.

Notice that $\check{G}_\infty^{(0)}$ is the set of all $(\alpha, t, \beta) \in \check{G}_\infty$ such that $\alpha = \beta$. In order to prove the $\check{G}_\infty$ is r-discrete, we need to show that $\check{G}_\infty^{(0)}$ is open in $\check{G}_\infty$. Let $(\alpha_\lambda, t_\lambda, \beta_\lambda) \in \check{G}_\infty$ be a net converging to $(\alpha, t, \alpha) \in \check{G}_\infty^{(0)}$. Then there exists λ_0 such that for $\lambda \geq \lambda_0$, $(\alpha_\lambda)_0 = \alpha_0$, $(\beta_\lambda)_0 = \alpha_0$ and $\alpha_\lambda(n) = \beta_\lambda(n)$ for $n > 0$. In other words, $\alpha_\lambda = \beta_\lambda$ and hence $(\alpha_\lambda, t_\lambda, \beta_\lambda) \in \check{G}_\infty^{(0)}$ for $\lambda \geq \lambda_0$. $\square$

Example 5.6 (In general $\check{G}_\infty$ is not étale.)**.** Let $\mathcal{T}$ be the interval $[0, 1]$. We define the ordered cover refinement V as follows: let $V_1^{(0)} = \mathcal{T}$, $V_2^{(0)} = (\frac{1}{2}, 1]$, and for every $k > 0$ let $V^{(k)}$ be the single set $V_1^{(k)} = \mathcal{T}$. In order to show that $\check{G}_\infty$ is not étale, it suffices to prove that the the range map $r : \check{G}_\infty \to \check{G}_\infty^{(0)}$ is not open. Take the point $x = (111\overline{1}, \frac{1}{2}, 211\overline{1}) \in \check{G}_\infty$, and let $\mathcal{O} = \check{Z}_{1,2,(\frac{1}{4},\frac{3}{4}),0}$ be an open neighborhood of

x. Any point $z \in \mathcal{O}$ must be of the form $(111\overline{1}, t, 211\overline{1})$, where $t \in [\frac{1}{2}, \frac{3}{4})$. Therefore $r(\mathcal{O}) = \{(111\overline{1}, t, 111\overline{1}) \mid t \in [\frac{1}{2}, \frac{3}{4})\}$, which is not an open set.

Example 5.7 (In general G_∞ is not locally compact). Let $\mathcal{T} = [0, 1]$, and for every $k \geq 0$ let $V_0^{(k)} = \mathcal{T}$, $V_1^{(k)} = (1/2, 1]$. Recall that $\mathcal{T} * \mathcal{R}(\Omega)$ is metrizable by Proposition 3.5, hence so are G_∞ and $\check{G}_\infty$. We show that the point $x = (000\overline{0}, \frac{1}{2}, 000\overline{0})$, does not have a compact neighborhood. In fact, if $\mathcal{O}$ is a compact neighborhood of x, then it contains an open neighborhood of x of the form $\check{Z}_{00\ldots0,00\ldots0,U,n} \cap G_\infty = \{(\alpha, t, \alpha) \in G_\infty : t \in U, \ \alpha|n \equiv 0\}$ for some $n \in \mathbb{N}$ and U an open subset of $\mathcal{T}$. Consider the sequence $x_k = (00\ldots0111\overline{1}, \frac{1}{2} + \frac{1}{k}, 00\ldots0111\overline{1})$ where there are exactly n zeros, followed by infinitely many ones. For k large enough, $x_k \in \check{Z}_{00\ldots0,00\ldots0,U,n} \cap G_\infty$, and x_k converges in $\check{G}_\infty$ to $(00\ldots0111\overline{1}, 1/2, 00\ldots0111\overline{1})$, however this point is not in G_∞. It follows that this is a sequence in $\mathcal{O}$ without a convergent subsequence in G_∞.

Lemma 5.8. *The map $\pi_N : \check{G}_\infty \to \check{G}_N$ given by $\pi_N(\alpha, t, \beta) = (\alpha|N, t, \beta|N)$ is a surjective continuous mapping and a groupoid homomorphism.*

Proof. It is easy to check that π_N is a well-defined continuous groupoid homomorphism. In order to show that it is surjective, let $(x, t, y) \in \check{G}_N$. Since for every n, $V^{(n)}$ is a cover of $\mathcal{T}$, there exists a sequence $\gamma = \{\gamma_k\} \in \Omega$ such that for any $k > N$, $V_{\gamma_k}^{(k)}$ contains t. Define $\alpha, \beta \in \Omega$ by setting $\alpha|N = x$, $\beta|N = y$ and $\alpha_n = \beta_n = \gamma_n$ for $n > N$. Clearly $(\alpha, \beta) \in \mathcal{R}(\Omega)$. Moreover, denote $O = V_{\gamma_{N+1}}^{(N+1)}$ and $A = W_x$. We then have that $\overline{A} = \overline{W_x} = \bigcap_{n=0}^{N} \overline{W_{\alpha|n}}$. Since $t \in O \cap \overline{A}$, we have by Remark 4.10, that $t \in \overline{O \cap A} = \overline{V_{\gamma_{N+1}}^{(N+1)} \cap W_{\alpha|N}} = \overline{W_{\alpha|N+1}}$. Repeating this reasoning proves that for every M, $t \in \overline{W_{\alpha|M}} \cap \overline{W_{\beta|M}}$. Thus $(\alpha, t, \beta) \in \check{G}_\infty$ and satisfies $\pi_N(\alpha, t, \beta) = (x, t, y)$. $\square$

We omit the similar proof of the following lemma.

Lemma 5.9. *For every $n \leq m$, define a map $\pi_n^m : \check{G}_m \to \check{G}_n$ by $\pi_n^m(\alpha, t, \beta) = (\alpha|n, t, \beta|n)$. The maps π_n^m are surjective continuous groupoid homomorphisms.*

Lemma 5.10. *G is an open subgroupoid of $\check{G}_\infty$.*

Proof. We showed in Lemma 4.9 that G is a subgroupoid of $\check{G}_\infty$. From the definition of G it follows that
$$G = \bigcup_{n \in \mathbb{N}} [\pi_n^{-1}(G_n) \cap \mathcal{Y}_n].$$
For every $n \in \mathbb{N}$, G_n is open in $\check{G}_n$ and π_n is continuous, hence $\pi_n^{-1}(G_n)$ is an open subset of $\check{G}_\infty$. Since $\mathcal{Y}_n$ is an open set for all n, we conclude that G is an open subset of $\check{G}_\infty$. $\square$

Theorem 5.11. *G is a locally compact, metrizable, second countable, Hausdorff, principal groupoid. Furthermore, it is étale and amenable.*

Proof. As a subgroupoid of $\check{G}_\infty$, G inherits the properties of being second countable, Hausdorff and metrizable, and by Proposition 4.9 it is principal. Furthermore, by Lemma 5.10, G is open in $\check{G}_\infty$ and the latter is locally compact, hence G is locally compact.

In order to prove the G is étale, first we must show that $G^{(0)}$ is open in G. This follows immediately from the observation that $G^{(0)} = G \cap \check{G}_\infty^{(0)}$, and both are open subsets of $\check{G}_\infty$ by Lemma 5.10 and since $\check{G}_\infty$ is r-discrete. Next we prove that r is a local homeomorphism by showing that is is open. Since G has a basis of open G-sets, it follows that r is a local homeomorphism if and only if it is open.

Let $\mathcal{O}$ be a non-empty open subset of G, and let $x = (\alpha, t, \beta) \in \mathcal{O}$. Let $Z_x = \check{Z}_{\alpha,\beta,U,n} \cap G$ be an open neighborhood of x inside $\mathcal{O}$. Since $x \in G$, there exists $k \in \mathbb{N}$ such that

$t \in W_{\alpha|k} \cap W_{\beta|k}$ and $\alpha_i = \beta_i$ for $i > k$. We may assume without loss of generality that $n > k$, and also that $U \subseteq W_{\alpha|k} \cap W_{\beta|k}$. We claim that $r(Z_x) = \check{Z}_{\alpha,\alpha,U,n} \cap G$. It is clear that $r(Z_x) \subseteq \check{Z}_{\alpha,\alpha,U,n} \cap G$. Fix $(\gamma, s, \gamma) \in \check{Z}_{\alpha,\alpha,U,n} \cap G$. Then we have that $s \in U$, $s \in \overline{W_{\gamma|j}}$ for all j, and

$$\gamma = (\alpha_0, \alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n, \gamma_{n+1}, \gamma_{n+2}, \ldots).$$

We define

$$\delta = (\beta_0, \beta_1, \ldots, \beta_k, \alpha_{k+1}, \ldots, \alpha_n, \gamma_{n+1}, \gamma_{n+2}, \ldots),$$

keeping in mind that $\alpha_{k+1}, \ldots, \alpha_n = \beta_{k+1}, \ldots, \beta_n$. Obviously $r(\gamma, s, \delta) = (\gamma, s, \gamma)$. The fact that $(\gamma, s, \delta) \in \check{Z}_{\alpha,\beta,U,n} \cap G = Z$ follows from the following observations:

(1) $s \in U \subseteq W_{\alpha|k} \cap W_{\beta|k} = W_{\gamma|k} \cap W_{\delta|k}$, therefore $(\gamma|k, s, \delta|k) \in G_k$.
(2) $\gamma_i = \delta_i$ for all $i > k$.
(3) For all j, $s \in \overline{W_{\gamma|j}}$.
(4) For all j, $s \in \overline{W_{\delta|j}}$: for $j \leq k$ we have that $s \in U \subseteq W_{\beta|j} = W_{\delta|j}$, and for $j > k$, by Remark 4.10, we have that $s \in U \cap \overline{W_{\gamma|j}} \subseteq \overline{U \cap W_{\gamma|j}} \subseteq \overline{W_{\delta|j}}$.

We conclude that $r(Z_x) = \check{Z}_{\alpha,\alpha,U,n} \cap G$ is an open set, which is clearly contained in $r(\mathcal{O})$. It follows that $r(\mathcal{O})$ is open, since $r(\mathcal{O}) = \bigcup_x r(Z_x)$.

Finally, in order to prove amenability of G we employ Proposition 5.1.1 of [ADR00], which states that if H is a locally closed subgroupoid of an amenable locally compact groupoid H' and the source and range maps of H are open then H is amenable. From Lemma 5.10 we have that G is open in $\check{G}_\infty$, therefore there exists an open set A in $\mathcal{T} * \mathcal{R}(\Omega)$ such that $G = A \cap \check{G}_\infty$; since $\check{G}_\infty$ is closed in $\mathcal{T} * \mathcal{R}(\Omega)$, we conclude that G is the intersection of open and closed subsets of $\mathcal{T} * \mathcal{R}(\Omega)$, hence it is locally closed (see section I.3.3 of [Bou89]). We have already seen that G is étale, therefore its range and source maps are open. By Proposition 3.9, $\mathcal{T} * \mathcal{R}(\Omega)$ is amenable, hence we conclude that G is amenable. □

6. Maps between Groupoid C*-Algebras

We present two propositions which are general, both of the same nature: under certain assumptions on two groupoids, we obtain an isometric *-homomorphism between the corresponding groupoid C^*-algebras. The composition of the these maps will play a key role in our main theorem.

Proposition 6.1. *Let Z and Q be locally compact Hausdorff principal étale groupoids endowed with the respective counting Haar systems, and let $\sigma \in Z^2(Q, \mathbb{T})$. Let $\pi : Z \to Q$ be a surjective continuous proper groupoid homomorphism, and suppose Q acts on Z to the left. Suppose the action is free, transitive and that π is equivariant with respect to the action. Then the map $\pi^* : C_c(Q, \sigma) \to C_c(Z, \pi^* \sigma)$ given by $\pi^* f = f \circ \pi$ is an isometric *-homomorphism with respect to reduced C^*-norms, which therefore extends to the reduced C^*-algebras.*

Proof. The map is well-defined because π is continuous and proper, hence given $f \in C_c(Q)$, the function $f \circ \pi$ is also continuous and compactly supported on Z.

It is straightforward to check that π^* is linear and *-preserving. We now show that it is multiplicative. Let $f, g \in C_c(Q, \sigma)$ and fix $x \in Z$.

$$(\pi^* f * \pi^* g)(x) = \int_Z \pi^* f(xy) \pi^* g(y^{-1}) \pi^* \sigma(xy, y^{-1}) d\lambda_Z^{d(x)}(y)$$

Recall that λ^u is a measure supported on Z^u and that $\pi^*\sigma(x, y) = \sigma(\pi(x), \pi(y))$ by the definition of a pullback cocycle. Thus, since π is a groupoid homomorphism, we obtain:

$$(\pi^*f * \pi^*g)(x) = \int_{Z^{d(x)}} f(\pi(x)\pi(y))g(\pi(y)^{-1})\sigma(\pi(x)\pi(y), \pi(y)^{-1})d\lambda_Z^{d(x)}(y)$$

We state the next argument as a lemma.

Lemma 6.2. *Suppose $u \in Z^{(0)}$ and let $v = \pi(u)$. Then $\pi : Z_u \to Q_v$ is a bijection.*

Proof. Take $x \in Z_u$. By definition $d(x) = u = d(u)$. Therefore, since the action is transitive, there exists $\gamma \in Q$ such that $x = \gamma \cdot u$.

Since the action is free, γ is determined uniquely. Indeed, suppose $\gamma \cdot u = \widetilde{\gamma} \cdot u$. Then $\gamma^{-1}\gamma \cdot u = \gamma^{-1}\widetilde{\gamma} \cdot u$, so $d(\gamma) \cdot u = \gamma^{-1}\widetilde{\gamma} \cdot u$, and since $d(\gamma) = r(u)$ and $r(u) \cdot u = u$ we see that $u = \gamma^{-1}\widetilde{\gamma} \cdot u$. Freeness now implies that $\gamma^{-1}\widetilde{\gamma} = d(\gamma^{-1}\widetilde{\gamma}) = r(u) = d(\gamma)$. Thus $\gamma\gamma^{-1}\widetilde{\gamma} = \gamma d(\gamma) = \gamma$, so $r(\gamma)\widetilde{\gamma} = \gamma$. But $r(\gamma) = r(\gamma \cdot u) = r(\widetilde{\gamma} \cdot u) = r(\widetilde{\gamma})$, hence $r(\widetilde{\gamma})\widetilde{\gamma} = \gamma$, proving $\widetilde{\gamma} = \gamma$.

¿From the equivariance of π with respect to the action, it follows that

$$\pi(x) = \pi(\gamma \cdot u) = \gamma\pi(u) = \gamma v.$$

In particular this shows that $d(\gamma) = v$, hence $\gamma \in Q_v$ and $\gamma v = \gamma$. We conclude that $\pi(x) = \gamma$ for a unique $\gamma \in Q_v$, and therefore we have a bijection. $\square$

We return to the proof of the proposition. By considering inverses, we have a bijection $\pi : Z^u \to Q^v$. So the map $\pi : Z^{d(x)} \to Q^{\pi(d(x))}$ is a bijection. Now $\pi(d(x)) = \pi(xx^{-1}) = \pi(x)\pi(x^{-1}) = d(\pi(x))$, so $\pi : Z^{d(x)} \to Q^{d(\pi(x))}$ is a bijection and per force it is measure preserving with respect to the counting Haar systems. It follows that:

$$\begin{aligned}(\pi^*f * \pi^*g)(x) &= \int_{Q^{d(\pi(x))}} f(\pi(x)z)g(z^{-1})\sigma(\pi(x)z, z^{-1})d\lambda_Q^{d(\pi(x))}(z) \\ &= (f * g)(\pi(x)) \\ &= \pi^*(f * g)(x)\end{aligned}$$

Thus π^* is multiplicative, and hence a $*$-homomorphism.

We prove next that π^* is an isometry. Let $v \in Z^{(0)}$ and let δ_v be the probability measure on $Z^{(0)}$ concentrated on v. Let ϵ_v be the probability measure on $Q^{(0)}$ concentrated on $\pi(v)$. We claim that the following equality holds:

(6.2.1) $\|\operatorname{Ind}_{\delta_v}^Z(\pi^*f)\| = \|\operatorname{Ind}_{\epsilon_v}^Q(f)\|$

By definition, $\operatorname{Ind}_{\delta_v}^Z(\pi^*f)$ acts on $L^2(Z, \nu^{-1})$, where in this case $\nu = \int \lambda_Z^u d\delta_v(u) = \lambda_Z^v$, hence ν^{-1} is counting measure on Z_v and $L^2(Z, \nu^{-1}) \simeq \ell^2(Z_v)$. Similarly, $\operatorname{Ind}_{\epsilon_v}^Q(f)$ acts on $\ell^2(Q_{\pi(v)})$.

The Haar system of Z is given by counting measures, therefore we can write for $\xi \in \ell^2(Z_v)$,

$$[\operatorname{Ind}_{\delta_v}^Z(\pi^*f)\xi](x) = \sum_{y \in Z^v} \pi^*f(xy)\xi(y^{-1})\pi^*\sigma(xy, y^{-1})$$

Since the map $\pi : Z_v \to Q_{\pi(v)}$ is a bijection, we have a unitary operator $W : \ell^2(Q_{\pi(v)}) \to \ell^2(Z_v)$ given by $W\psi = \psi \circ \pi$. We now show that

$$\operatorname{Ind}_{\delta_v}^Z(\pi^*f)W = W\operatorname{Ind}_{\epsilon_v}^Q(f).$$

Indeed, if $x \in Z_v$,

$$[\text{Ind}_{\delta_v}^Z (\pi^* f) W \psi](x) = \sum_{y \in Z^v} \pi^* f(xy)[W \psi](y^{-1}) \pi^* \sigma(xy, y^{-1})$$

$$= \sum_{y \in Z^v} f(\pi(x)\pi(y)) \psi(\pi(y)^{-1}) \sigma(\pi(x)\pi(y), \pi(y)^{-1})$$

$$= \sum_{z \in Q^{\pi(v)}} f(\pi(x)z) \psi(z^{-1}) \sigma(\pi(x)z, z^{-1})$$

$$= [\text{Ind}_{\epsilon_v}^Q (f) \psi](\pi(x))$$

$$= [W \text{Ind}_{\epsilon_v}^Q (f) \psi](x)$$

Therefore the operators $\text{Ind}_{\delta_v}^Z (\pi^* f)$ and $\text{Ind}_{\epsilon_v}^Q (f)$ are unitarily equivalent, so they share the same norm. Thus we have obtained equation (6.2.1).

Finally, since every unit point mass on Q is of the form ϵ_v for some $v \in Z$, we have:

$$\|\pi^* f\|_{C_r^*(Z, \pi^* \sigma)} = \sup\{\| \text{Ind}_{\delta_v}^Z (\pi^* f)\| : v \in Z^{(0)}\}$$

$$= \sup\{\| \text{Ind}_{\epsilon_v}^Q (f)\| : v \in Z^{(0)}\}$$

$$= \sup\{\| \text{Ind}_{\mu}^Q (f)\| : \mu \text{ is a unit point mass on } Q^{(0)}\}$$

$$= \|f\|_{C_r^*(Q, \sigma)}$$

$\square$

Proposition 6.3. *Let R be a locally compact, Hausdorff, étale amenable groupoid admitting as left Haar system the counting measures system and let $\sigma \in Z^2(R, \mathbb{T})$. Suppose that Z is an open subgroupoid of R endowed with the restriction Haar system, and let $\rho : C_c(Z, \sigma) \hookrightarrow C_c(R, \sigma)$ be the natural inclusion obtained by extending functions as identically zero outside Z. Then ρ is an isometric $*$-homomorphism, and thus extends to the groupoid C^*-algebras.*

Proof. It is clear that the map ρ is well-defined: let $f \in C_c(Z)$ with $K = \text{supp} f$ compact. The set K is closed in R since R is Hausdorff, and f is zero on ∂K. Since Z is open, the extension to R is continuous and compactly supported.

It is straightforward to check that ρ is linear and $*$-preserving. We now show that ρ is multiplicative. Let $f, g \in C_c(Z)$. Since $\text{supp} \rho(f) \subseteq Z$ and $\text{supp} \rho(g) \subseteq Z$, it follows that $\text{supp}(\rho(f) * \rho(g)) \subseteq Z$ because Z is closed under the groupoid operations of R. Thus it suffices to verify that $[\rho(f) * \rho(g)](x) = [\rho(f * g)](x)$ for all $x \in Z$. Fix $x \in Z$.

$$[\rho(f) * \rho(g)](x) = \int_R \rho(f)(xy) \rho(g)(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y)$$

$$= \int_Z \rho(f)(xy) \rho(g)(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y)$$

$$= \int_Z f(xy) g(y^{-1}) \sigma(xy, y^{-1}) d\lambda^{d(x)}(y)$$

$$= (f * g)(x)$$

$$= [\rho(f * g)](x)$$

where we used the fact that the Haar system of Z is the restriction of the Haar system of R.

We claim that ρ is an isometry:

$$\|\rho(f)\|_{C^*(R, \sigma)} = \|f\|_{C^*(Z, \sigma)}, \qquad \forall f \in C_c(Z, \sigma).$$

Observe that bounded representations of $C_c(R, \sigma)$ give rise to bounded representations of $C_c(Z, \sigma)$ by composition with ρ and taking cut-downs to ensure non-degeneracy. Thus we have

$$\|\rho(f)\|_{C^*(R,\sigma)} \leq \|f\|_{C^*(Z,\sigma)}, \qquad \forall f \in C_c(Z, \sigma).$$

In order to prove the opposite inequality, recall that Z is an open subgroupoid of R amenable, hence it is also amenable (see Proposition 5.1.1 in [ADR00]). Thus it suffices to convert to the reduced norms, and show that

$$\|\rho(f)\|_{C_r^*(R,\sigma)} \geq \|f\|_{C_r^*(Z,\sigma)}, \qquad \forall f \in C_c(Z, \sigma).$$

So let δ be a probability measure on $Z^{(0)}$ concentrated on a unit v. We will abuse notation slightly and denote also by δ the probability measure on $R^{(0)}$ supported on v. By definition, $\mathrm{Ind}_\delta^R(f)$ acts on $L^2(R, \nu^{-1})$, where in this case $\nu = \int \lambda^u d\delta(u) = \lambda^v$, hence ν^{-1} is counting measure on R_v and $L^2(R, \nu^{-1}) \simeq \ell^2(R_v)$. Furthermore, we can write for $\psi \in \ell^2(R_v)$,

$$[\mathrm{Ind}_\delta^R(\rho(f))\psi](x) = \sum_{y \in R^v} \rho(f)(xy)\psi(y^{-1})\sigma(xy, y^{-1}).$$

The operator $\mathrm{Ind}_\delta^Z(f)$ acts analogously on $\ell^2(Z_v)$.

Consider the inclusion operator $W : \ell^2(Z_v) \to \ell^2(R_v)$ where $W\xi$ is the extension of ξ as identically zero outside of Z_v. Then W is an isometry and $W^* : \ell^2(R_v) \to \ell^2(Z_v)$ is the restriction operator $W^*\psi = \psi \restriction_{Z_v}$. Now observe that given $\psi \in \ell^2(Z_v)$ and $x \in Z_v$,

$$
\begin{aligned}
(\mathrm{Ind}_\delta^R(\rho(f))W\psi)(x) &= \sum_{y \in R^v} \rho(f)(xy)W\psi(y^{-1})\sigma(xy, y^{-1}) \\
&= \sum_{y \in Z^v} \rho(f)(xy)\psi(y^{-1})\sigma(xy, y^{-1}) \\
&= \sum_{y \in Z^v} f(xy)\psi(y^{-1})\sigma(xy, y^{-1}) \\
&= (\mathrm{Ind}_\delta^Z(f)\psi)(x) \\
&= W(\mathrm{Ind}_\delta^Z(f)\psi)(x)
\end{aligned}
$$

If $x \notin Z_v$ then $xy \notin Z$, so $\rho(f)(xy) = 0$ and the whole expression is zero. Thus for any $\psi \in \ell^2(Z_v)$ we have

$$(\mathrm{Ind}_\delta^R(\rho(f))W\psi) = W(\mathrm{Ind}_\delta^Z(f)\psi).$$

Since W corresponds to extension with zero outside of Z_v and the norm is the ℓ^2 norm, we conclude that:

$$\| \mathrm{Ind}_\delta^R(\rho(f))W\psi\| = \|W \mathrm{Ind}_\delta^Z(f)\psi\| = \| \mathrm{Ind}_\delta^Z(f)\psi\|$$

This allows us to obtain the following inequality:

$$
\begin{aligned}
\| \mathrm{Ind}_\delta^R(\rho(f))\| &= \sup\{\| \mathrm{Ind}_\delta^R(\rho(f))\psi\| : \psi \in \ell^2(R_v), \|\psi\| \leq 1\} \\
&\geq \sup\{\| \mathrm{Ind}_\delta^R(\rho(f))W\psi\| : \psi \in \ell^2(Z_v), \|\psi\| \leq 1\} \\
&= \sup\{\| \mathrm{Ind}_\delta^Z(f)\psi\| : \psi \in \ell^2(Z_v), \|\psi\| \leq 1\} \\
&= \| \mathrm{Ind}_\delta^Z(f)\|
\end{aligned}
$$

Therefore we have:

$$\|\rho(f)\|_{C_r^*(R,\sigma)} = \sup\{\|\operatorname{Ind}_\delta^R(\rho(f))\| : \delta \text{ is a unit point mass on } R^{(0)}\}$$
$$\geq \sup\{\|\operatorname{Ind}_\delta^R(\rho(f))\| : \delta \text{ is a unit point mass on } Z^{(0)}\}$$
$$\geq \sup\{\|\operatorname{Ind}_\delta^Z(f)\| : \delta \text{ is a unit point mass on } Z^{(0)}\}$$
$$= \|f\|_{C_r^*(Z,\sigma)}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7. The C^*-algebra of G

Our main theorem asserts that under a certain assumption regarding the cocycles, for every n there is an isometric $*$-homomorphism

$$\varphi_n : C_r^*(G_n, \sigma_n) \to C^*(G, \sigma).$$

In order to state our theorem precisely, we require the following definition.

Definition 7.1. Given a cocycle $\sigma_n \in Z^2(G_n, \mathbb{T})$, we will say that it extends to $\check{G}_n$ if there exists $\check{\sigma}_n \in Z^2(\check{G}_n, \mathbb{T})$ such that $\sigma_n = \check{\sigma}_n$ on $G_n^{(2)}$. Observe that such an extension is unique if it exists. We denote by $Z_{\text{ext}}^2(G_n, \mathbb{T})$ the subgroup of cocycles which can be extended to $\check{G}_n$.

A continuous 2-cocycle $\sigma_n \in Z^2(G_n, \mathbb{T})$ which extends to $\check{G}_n$, will also extend to a continuous 2-cocycle $\sigma \in Z^2(G, \mathbb{T})$, by means of pullback: Recall the maps $\pi_N : \check{G}_\infty \to \check{G}_N$ given by $\pi_N(\alpha, t, \beta) = (\alpha|N, t, \beta|N)$, introduced in Lemma 5.8. When restricted to G, the maps $\pi_N : G \to \check{G}_N$ remain continuous groupoid homomorphisms, although they are no longer surjective. Nevertheless, if $\check{\sigma}_n \in Z^2(\check{G}_n, \mathbb{T})$ then $\sigma = \pi_n^* \check{\sigma}_n$ is a continuous 2-cocycle in $Z^2(G, \mathbb{T})$.

We can now state our main theorem.

Theorem 7.2. *For every $n \geq 0$ and $\sigma_n \in Z_{\text{ext}}^2(G_n, \mathbb{T})$, denote by $\check{\sigma}_n$ its extension in $Z^2(\check{G}_n, \mathbb{T})$. Let $\sigma = \pi_n^* \check{\sigma}_n \in Z^2(G, \mathbb{T})$ be the pullback cocycle obtained from the map $\pi_n : G \to \check{G}_n$. Then the map $\varphi_n : C_c(G_n, \sigma_n) \to C_c(G, \sigma)$ given by*

$$[\varphi_n(f)](x) = \begin{cases} f(\pi_n(x)) & x \in \pi_n^{-1}(G_n) \cap \mathcal{Y}_n \\ 0 & \text{otherwise} \end{cases}$$

is an isometric $*$-homomorphism, which extends to an isometric $*$-homomorphism $\varphi_n : C_r^*(G_n, \sigma_n) \to C^*(G, \sigma)$.

Proof. Fix $n \in \mathbb{N}$. We will apply Propositions 6.1 and 6.3 in succession.

Let $Q = G_n$, $Z = \pi_n^{-1}(G_n) \cap \mathcal{Y}_n$, and $R = G$. Notice that $Z \subseteq G$ and consider the following maps:

- $\pi : Z \to G_n$ given by the restriction of π_n to Z, and its pullback

$$\pi^* : C_c(G_n, \sigma_n) \to C_c(Z, \pi^* \sigma_n) = C_c(Z, \sigma).$$

- $\rho : C_c(Z, \sigma) \to C_c(G, \sigma)$ which extends functions as identically zero outside of Z.

We start with the map ρ. Notice first that Z is an open subgroupoid of G. We will endow Z with the restriction counting Haar system. G satisfies all the conditions of Proposition 6.3 by Theorem 5.11. Therefore the map ρ is an isometric $*$-homomorphism.

We now turn to the map π^*. Inheriting properties from G of which it is an open subgroupoid, Z as a groupoid is étale, principal, locally compact and Hausdorff.

The map π is clearly a surjective continuous groupoid homomorphism. To see that it is proper, let $K \subseteq G_n$ be compact. It will be useful to refer not only to the projection

$\pi_n : Z \to G_n$, but also to the original on $\check{G}_\infty$, which we temporarily denote by $\check{\pi}_n : \check{G}_\infty \to \check{G}_n$ for distinction. Notice that

$$\pi^{-1}(K) = \pi_n^{-1}(K) \cap \mathcal{Y}_n = \check{\pi}_n^{-1}(K) \cap \mathcal{Y}_n.$$

K is compact hence closed in $\check{G}_n$, since $\check{G}_n$ is Hausdorff. Thus $\check{\pi}_n^{-1}(K)$ is closed in $\check{G}_\infty$ since $\check{\pi}_n$ is continuous by Lemma 5.8. Since $\mathcal{Y}_n$ is compact in $\check{G}_\infty$, it follows that $\pi^{-1}(K)$ is a compact set.

Let G_n act on Z on the left with respect to the map $r_{act} : Z \to G_n^{(0)}$ given by $r_{act}(x) = \pi(r_Z(x))$ as follows. An element $\gamma = (i, t, j) \in G_n$ can act on $x = (\alpha, s, \beta) \in Z$ if and only if $d_{G_n}(\gamma) = r_{act}(x)$, that is to say $t = s$ and $j = \alpha|n$. In that case,

$$\gamma \cdot x := (i_0 i_1 \ldots i_n \alpha_{n+1} \alpha_{n+2} \ldots, t, \beta)$$

It is clear that the map r_{act} is a surjection. The action map $(\gamma, x) \mapsto \gamma \cdot x$ is well-defined: $\gamma = (i, t, j)$ is in G_n, so $t \in W_i \cap W_j$. Since $x = (\alpha, t, \beta) \in Z$, we have that $\alpha_k = \beta_k$ for $k > n$, $t \in W_{\beta|n}$ and $t \in \overline{W_{\beta|k}}$ for all k. Thus by Remark 4.10 we have that $t \in \overline{W_i \cap W_{\beta|k}}$ for all k. In particular, if we set $\delta = i_0 i_1 \ldots i_n \beta_{n+1} \ldots$, then we have that for every k, $t \in \overline{W_{\delta|k}}$. We conclude that the element $\gamma \cdot x = (\delta, t, \beta)$ indeed belongs to Z.

It is a straightforward verification that this is a left action, i.e. that axioms (1) - (3) of the definition are satisfied. This action is free because if $\gamma \cdot x = x$ and $x = (\alpha, t, \beta)$ then we must have $\gamma = (\alpha|n, t, \alpha|n) = d_{G_n}(\gamma) = r_{act}(x)$. In order to see that it is transitive, suppose $y \in Z$ and $d_Z(y) = d_Z(x)$. Then we can write $y = (j_1 j_2 \ldots j_n \beta_{n+1} \beta_{n+2} \ldots, t, \beta)$. Since both $x, y \in Z$, we have that $t \in W_{\alpha|n}$ and $t \in W_j$. Therefore, if we set $\gamma = (\alpha|n, t, j)$ we have that γ is an element of G_n and $\gamma \cdot y = x$. Last but not least, π is equivariant with respect to the action. Indeed, if $(\gamma, x) \in G_n * Z$ then $d_{G_n}(\gamma) = r_{act}(x) = \pi(r_Z(x))$. It is straightforward to verify that $\pi(r_Z(x)) = r_{G_n}(\pi(x))$, therefore $d_{G_n}(\gamma) = r_{G_n}(\pi(x))$. It is obvious that $\pi(\gamma \cdot x) = \gamma \pi(x)$.

Having verified all the conditions of Proposition 6.1, we conclude that π^* is an isometric $*$-homomorphism.

Finally, we observe that φ_n is precisely $\rho \circ \pi^*$. Thus $\varphi_n : C_c(G_n, \sigma_n) \to C_c(G, \sigma)$ is an isometric $*$-homomorphism. It follows from the general theory of C^*-algebras that φ_n extends to the C^*-completions. $\qquad\square$

8. $C^*(G, \sigma)$ AS A GENERALIZED DIRECT LIMIT

We first consider the most trivial case where $\mathcal{T}$ is a single point. Open covers of $\mathcal{T}$ are merely repetitions of the singleton set, and are determined by their cardinality. It is easy to see that in this case G, G_∞ and $\check{G}_\infty$ all coincide with each other and with $\mathcal{T} * \mathcal{R}(\Omega)$, which is in turn isomorphic to $\mathcal{R}(\Omega)$. Therefore $C^*(G)$ is a UHF C^*-algebra (The cohomology of $\mathcal{T}$ is of course trivial, so there are no cocycles involved). Moreover, the algebras $C^*(G_n)$ in this case are matrix algebras, and $C^*(G) = \varinjlim C^*(G_n)$.

We would also have $C^*(G)$ as a direct limit in the case where $\mathcal{T}$ is a finite (discrete) set of points, where $C^*(G)$ can be seen to be an AF C^*-algebra. However, in general this is not the case. In order to be precise, we require the following definition.

Definition 8.1. Let $n \geq 0$ and take $\sigma_n \in Z_{ext}^2(G_n, \mathbb{T})$. Denote by $\check{\sigma}_n$ its extension in $Z^2(\check{G}_n, \mathbb{T})$. Let $m > n$, denote by $\check{\sigma}_m \in Z^2(\check{G}_m, \mathbb{T})$ the pullback cocycle obtained from the map $\pi_n^m : \check{G}_m \to \check{G}_n$ (see Lemma 5.9), and let $\sigma_m \in Z_{ext}^2(G_m, \mathbb{T})$ be its restriction to G_m. We will then say that σ_n and σ_m are **compatible cocycles**.

Now fix $n_0 \geq 0$. Notice that if $n \leq m \leq k$ then $\pi_n^k = \pi_n^m \circ \pi_m^k$ and furthermore, $\pi_n = \pi_n^m \circ \pi_m$. It follows that we can extend this notion to define a sequence of compatible cocycles $\{\sigma_n\}_{n \geq n_0}$. Moreover, there is a well-defined cocycle $\sigma \in Z^2(G, \mathbb{T})$ which is

simultaneously the pullback of all the cocycles $\{\check{\sigma}_n\}_{n \geq n_0}$ with respect to the maps $\pi_n : G \to \check{G}_n$. We will say that $\{\sigma_n\}_{n \geq n_0}$ is a **sequence of compatible cocycles with a limit cocycle** σ.

Let $\{\sigma_n\}_{n \geq n_0}$ be a sequence of compatible cocycles with a limit cocycle σ. In general, $C^*(G, \sigma)$ is *not* a direct limit of the sequence $C^*(G_n, \sigma_n)$ since we have no maps $C^*(G_n, \sigma_n) \to C^*(G_{n+1}, \sigma_{n+1})$. Moreover, $\bigcup_n(\varphi_n(C^*(G_n, \sigma_n)))$ is not dense inside $C^*(G, \sigma)$.

To see this, consider the following two points in G: $x = (\alpha, t, \alpha)$ and $y = (\beta, t, \beta)$, where $\alpha = \alpha_0\alpha_1 \ldots \alpha_n\alpha_{n+1}\alpha_{n+2} \ldots$ and $\beta = \alpha_0\alpha_1 \ldots \alpha_n\beta_{n+1}\beta_{n+2} \ldots$, and where $t \in W_{\alpha|n} = W_{\beta|n}$ but $t \notin W_{\alpha|n+1}$ and $t \notin W_{\beta|n+1}$. Thus $x, y \in \pi_n^{-1}(G_n) \cap \mathcal{Y}_n$ whereas $x, y \notin \pi_m^{-1}(G_n) \cap \mathcal{Y}_m$ for $m > n$. Therefore $\varphi_k(f)(x) = f(\pi_k(x)) = f(\pi_k(y)) = \varphi_k(f)(y)$ for any $k \leq n$ and any $f \in C_c(G_k, \sigma_k)$, and $\varphi_k(f)(x) = 0 = \varphi_k(f)(y)$ for any $k > n$ and any $f \in C_c(G_k, \sigma_k)$. Thus $\bigcup_n(\varphi_n(C^*(G_n, \sigma_n)))$ cannot separate x and y.

Despite this, we regard $C^*(G, \sigma)$ as a generalized direct limit of the sequence $C^*(G_n, \sigma_n)$. Our justification for this is that G satisfies the following minimality property. Given a subset S of $C^*(G, \sigma)$, we will denote

$$\operatorname{supp} S = \{x \in G \mid \exists f \in S \text{ such that } f(x) \neq 0\}.$$

Proposition 8.2. $G = \operatorname{supp} \bigcup_n(\varphi_n(C^*(G_n, \sigma_n)))$.

Proof. We claim that for every $x \in G$ there exists $n \in \mathbb{N}$ and $f \in C_c(G_n, \sigma_n)$ such that $\varphi_n(f)(x) \neq 0$. The proof of this is simple. Let $x = (\alpha, t, \beta) \in G$. There exists $n \in \mathbb{N}$ such that $x \in \pi_n^{-1}(G_n) \cap \mathcal{Y}_n$, i.e. $\pi_n(x) \in G_n$. Take $f \in C_c(G_n, \sigma_n)$ such that $f(\pi_n(x)) \neq 0$. Then clearly $\varphi_n(f)(x) \neq 0$. $\square$

Remark 8.3. *In* [MS89], *P. Muhly and B. Solel present a bijective correspondence between closed subsets of G and $C^*(G^{(0)})$-bimodules. In order to invoke their results, certain assumptions on the groupoid G are required. Our G has all the required properties, except one: G does not admit a cover by compact open G-sets. (Note that $\check{G}_\infty$ does admit such a cover - take the basis sets of the form $\check{Z}_{\alpha,\beta,\mathcal{T},n}$. These are compact since they can be written as $p_n^{-1}(\{(\alpha|n, \beta|n)\}) \cap \mathcal{Y}_n$.) Nevertheless, it may be that the results of Muhly and Solel remain valid without the compact open G-sets assumption. Should this be the case, we would have as a corollary of Proposition 8.2 the following statement: The $C^*(G^{(0)})$-bimodule generated by $\bigcup_n(\varphi_n(C^*(G_n, \sigma_n)))$ is $C^*(G, \sigma)$.*

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REFERENCES

[ADR00] Claire Anantharaman-Delaroche and Jean Renault, *Amenable groupoids*, Monographies de L'Enseignement Mathématique, vol. 36, L'Enseignement Mathématique, Geneva, 2000, With a foreword by Georges Skandalis and Appendix B by E. Germain.

[Bou89] Nicolas Bourbaki, *General topology. Chapters 1–4*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989, Translated from the French, Reprint of the 1966 edition.

[DD63] Jacques Dixmier and Adrien Douady, *Champs continus d'espaces hilbertiens et de C*-algèbres*, Bull. Soc. Math. France **91** (1963), 227–284.

[Dix63] Jacques Dixmier, *Champs continus d'espaces hilbertiens et de C*-algèbres. II*, J. Math. Pures Appl. (9) **42** (1963), 1–20.

[Kel75]   John L. Kelley, *General topology*, Springer-Verlag, New York, 1975, Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27.

[MS89]   Paul S. Muhly and Baruch Solel, *Subalgebras of groupoid C^*-algebras*, J. Reine Angew. Math. **402** (1989), 41–75.

[Muh]   Paul S. Muhly, *Coordinates in operator algebras*, to appear in CBMS lecture notes series.

[Ren80] Jean Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980.

[Res07] Pedro Resende, *Étale groupoids and their quantales*, Adv. Math. **208** (2007), no. 1, 147–209.

[RT85]   Iain Raeburn and Joseph L. Taylor, *Continuous trace C^*-algebras with given Dixmier-Douady class*, J. Austral. Math. Soc. Ser. A **38** (1985), no. 3, 394–407.

[RW98]   Iain Raeburn and Dana P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998.

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